

# DIFFERENTIATION OPERATOR FROM MODEL SPACES TO BERGMAN SPACES AND PELLER TYPE INEQUALITIES

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**ABSTRACT.** Given an inner function  $\Theta$  in the unit disc  $\mathbb{D}$ , we study the boundedness of the differentiation operator which acts from the model subspace  $K_\Theta = (\Theta H^2)^\perp$  of the Hardy space  $H^2$ , equipped with the *BMOA*-norm, to some radial-weighted Bergman space. As an application, we generalize Peller's inequality for Besov norms of rational functions  $f$  of degree  $n \geq 1$  having no poles in the closed unit disc  $\overline{\mathbb{D}}$ .

## 1. INTRODUCTION AND NOTATIONS

A well-known inequality by Vladimir Peller (see inequality (2.1) below) majorizes a Besov norm of any rational function  $f$  of degree  $n \geq 1$  having no poles in the closed unit disc  $\overline{\mathbb{D}} = \{\xi \in \mathbb{C} : |\xi| \leq 1\}$  in terms of its *BMOA*-norm and its degree  $n$ . The original proof of Peller is based on his description of Hankel operators in the Schatten classes. One of the aims of this paper is to give a short and direct proof of this inequality and extend it to more general radial-weighted Bergman norms. Our proof combines integral representation for the derivative of  $f$  (which come from the theory of model spaces) and the generalization of a theorem by E.M. Dyn'kin. The corresponding inequalities are obtained in terms of radial-weighted Bergman norms of the derivative of finite Blaschke products (of degree  $n = \deg f$ ), instead of  $n$  itself. The finite Blaschke products in question have the same poles as  $f$ . The study of radial-weighted Bergman norms of the derivatives of finite Blaschke products of degree  $n$  and their asymptotic as  $n$  tends to  $+\infty$  is of independent interest. A contribution to this topic, which we are going to exploit here, was given by J. Arazy, S.D. Fisher and J. Peetre.

Let  $\mathcal{P}_n$  be the space of complex analytic polynomials of degree at most  $n$  and let

$$\mathcal{R}_n^+ = \left\{ \frac{P}{Q} : P, Q \in \mathcal{P}_n, Q(\xi) \neq 0 \text{ for } |\xi| \leq 1 \right\}$$

be the set of rational functions of degree at most  $n$  with poles outside of the closed unit disc  $\overline{\mathbb{D}}$ . In this paper, we consider the norm of a rational function  $f \in \mathcal{R}_n^+$  in different spaces of analytic functions in the open unit disc  $\mathbb{D} = \{\xi : |\xi| < 1\}$ .

**1.1. Some Banach spaces of analytic functions.** We denote by  $\mathcal{H}ol(\mathbb{D})$  the space of all holomorphic functions in  $\mathbb{D}$ .

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1.1.1. *The Besov spaces  $B_p$ .* A function  $f \in \mathcal{H}ol(\mathbb{D})$  belongs to the Besov space  $B_p$ ,  $1 < p < \infty$ , if and only if

$$\|f\|_{B_p} = |f(0)| + \|f\|_{B_p}^* < +\infty,$$

where  $\|f\|_{B_p}^*$  is the seminorm defined by

$$\|f\|_{B_p}^* = \left( \int_{\mathbb{D}} (1 - |u|^2)^{p-2} |f'(u)|^p dA(u) \right)^{\frac{1}{p}},$$

$A$  being the normalized area measure on  $\mathbb{D}$ .

For the case  $0 < p \leq 1$  the definition of the Besov norm requires a modification:

$$\|f\|_{B_p} = \sum_{j=0}^{k-1} |f^{(j)}(0)| + \|f\|_{B_p}^*, \quad \|f\|_{B_p}^* = \left( \int_{\mathbb{D}} |f^{(k)}(u)|^p (1 - |u|^2)^{pk-2} dA(u) \right)^{\frac{1}{p}},$$

where  $k$  is the smallest positive integer such that  $pk > 1$ . We refer to [Pee, Tri, BeLo] for general properties of Besov spaces.

A function  $f \in \mathcal{H}ol(\mathbb{D})$  belongs to the space  $B_\infty$  (known as the Bloch space) if and only if  $\|f\|_{B_\infty} = |f(0)| + \sup_{z \in \mathbb{D}} |f'(z)| (1 - |z|) < \infty$ .

1.1.2. *The radial-weighted Bergman spaces  $A_p(w)$ .* The radial-weighted Bergman space  $A_p(w)$ ,  $1 \leq p < \infty$ , is defined as:

$$A_p(w) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \|f\|_{A_p(w)}^p = \int_{\mathbb{D}} w(|u|) |f(u)|^p dA(u) < \infty \right\},$$

where the weight  $w$  satisfies  $w \geq 0$  and  $\int_0^1 w(r) dr < \infty$ . The classical power weights  $w(r) = w_\alpha(r) = (1 - r^2)^\alpha$ ,  $\alpha > -1$ , are of special interest; in this case we put  $A_p(\alpha) = A_p(w_\alpha)$ . We refer to [HKZ] for general properties of weighted Bergman spaces.

1.1.3. *The spaces  $A_p^1(\alpha)$ .* A function  $f \in \mathcal{H}ol(\mathbb{D})$  belongs to the space  $A_p^1(\alpha)$ ,  $1 \leq p \leq +\infty$ ,  $\alpha > -1$ , if and only if

$$\|f\|_{A_p^1(\alpha)} = |f(0)| + \|f'\|_{A_p(\alpha)} < +\infty.$$

We also define the  $A_p^1(\alpha)$ -seminorm by  $\|f\|_{A_p^1(\alpha)}^* = \|f'\|_{A_p(\alpha)}$ . Note that the spaces  $B_p$  and  $A_p^1(p-2)$  coincide for  $1 < p < +\infty$ .

1.1.4. *The space  $BMOA$ .* There are many ways to define  $BMOA$ ; see [Gar, Chapter 6]. For the purposes of this paper we choose the following one: a function  $f \in \mathcal{H}ol(\mathbb{D})$  belongs to the  $BMOA$  space (of analytic functions of bounded mean oscillation) if and only if

$$\|f\|_{BMOA} = \inf \|g\|_{L^\infty(\mathbb{T})} < +\infty,$$

where the infimum is taken over all  $g \in L^\infty(\mathbb{T})$ ,  $\mathbb{T} = \{\xi : |\xi| = 1\}$  being the unit circle, for which the representation

$$f(\xi) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{g(u)}{u - \xi} du, \quad |\xi| < 1,$$

holds. Recall that  $BMOA$  is the dual space of the Hardy space  $H^1$  under the pairing

$$\langle f, g \rangle = \int_{\mathbb{T}} f(u) \overline{g(u)} du, \quad f \in H^1, g \in BMOA,$$

where this integral must be understood as the extension of the pairing acting on a dense subclass of  $H^1$ , see [Bae, p. 23].

## 1.2. Model spaces.

1.2.1. *General inner functions.* By  $H^p$ ,  $1 \leq p \leq \infty$ , we denote the standard Hardy spaces (see [Gar, Nik]). Recall that  $H^2$  is a reproducing kernel Hilbert space, with the kernel

$$k_\lambda(w) = \frac{1}{1 - \overline{\lambda}w}, \quad \lambda, w \in \mathbb{D},$$

known as the Szegő kernel (or the Cauchy kernel) associated with  $\lambda$ . Thus  $\langle f, k_\lambda \rangle = f(\lambda)$  for all  $f \in H^2$  and for all  $\lambda \in \mathbb{D}$ , where  $\langle \cdot, \cdot \rangle$  is the scalar product on  $H^2$ .

Let  $\Theta$  be an *inner function*, i.e.,  $\Theta \in H^\infty$  and  $|\Theta(\xi)| = 1$  a.e.  $\xi \in \mathbb{T}$ . We define the model subspace  $K_\Theta$  of the Hardy space  $H^2$  by

$$K_\Theta = (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

By the famous theorem of Beurling, these and only these subspaces of  $H^2$  are invariant with respect to the backward shift operator. We refer to [Nik] for the general theory of the spaces  $K_\Theta$  and their numerous applications.

For any *inner function*  $\Theta$ , the reproducing kernel of the model space  $K_\Theta$  corresponding to a point  $\xi \in \mathbb{D}$  is of the form

$$k_\lambda^\Theta(w) = \frac{1 - \overline{\Theta(\lambda)}\Theta(w)}{1 - \overline{\lambda}w}, \quad \lambda, w \in \mathbb{D},$$

that is  $\langle f, k_\lambda^\Theta \rangle = f(\lambda)$  for all  $f \in K_\Theta$  and for all  $\lambda \in \mathbb{D}$ .

1.2.2. *The case of finite Blaschke products.* From now on, for any  $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$ , we consider the finite Blaschke product

$$B_\sigma = \prod_{k=1}^n b_{\lambda_k},$$

where  $b_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda}z}$ , is the elementary Blaschke factor corresponding to  $\lambda \in \mathbb{D}$ . It is well known that if

$$\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\} \in \mathbb{D}^n,$$

where every  $\lambda_s$  is repeated according to its multiplicity  $n_s$ ,  $\sum_{s=1}^t n_s = n$ , then

$$K_{B_\sigma} = H^2 \ominus B_\sigma H^2 = \overline{\text{span}}\{k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq n_j - 1\},$$

where for  $\lambda \neq 0$ ,  $k_{\lambda, i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$  and  $k_\lambda = \frac{1}{1 - \overline{\lambda}z}$  is the standard Cauchy kernel at the point  $\lambda$ , whereas  $k_{0, i} = z^i$ . Thus the subspace  $K_{B_\sigma}$  consists of rational functions of the form  $p/q$ , where  $p \in \mathcal{P}_{n-1}$  and  $q \in \mathcal{P}_n$ , with the poles  $1/\overline{\lambda}_1, \dots, 1/\overline{\lambda}_n$  of corresponding multiplicities (including possible poles at  $\infty$ ). Hence, if  $f \in \mathcal{R}_n^+$  and  $1/\overline{\lambda}_1, \dots, 1/\overline{\lambda}_n$  are the poles of  $f$  (repeated according to multiplicities), then  $f \in K_{B_\sigma}$  with  $\sigma = (\lambda_1, \dots, \lambda_n)$ .

From now on, for two positive functions  $a$  and  $b$ , we say that  $a$  is dominated by  $b$ , denoted by  $a \lesssim b$ , if there is a constant  $C > 0$  such that  $a \leq Cb$ ; we say that  $a$  and  $b$  are comparable, denoted by  $a \asymp b$ , if both  $a \lesssim b$  and  $b \lesssim a$ .

## 2. MAIN RESULTS

**2.1. Main ingredients.** In 1980 V. Peller proved in his seminal paper [Pel1] that

$$(2.1) \quad \|f\|_{B_p} \leq c_p n^{\frac{1}{p}} \|f\|_{BMOA}$$

for any  $f \in \mathcal{R}_n^+$  and  $1 \leq p \leq +\infty$ , where  $c_p$  is a constant depending only on  $p$ . Later, this result was extended to the range  $p > 0$  independently and with different proofs by Peller [Pel2], S. Semmes [Sem] and also by A. Pekarskii [Pek1] who found a proof which does not use the theory of Hankel operators (see also [Pek2]).

The aim of the present article is:

- (1) study the boundedness of the differentiation operator from  $(K_\Theta, \|\cdot\|_{BMOA})$  to  $A_p(\alpha)$ ,  $1 < p < +\infty$ ,  $\alpha > -1$ , and
- (2) generalize Peller's result (2.1) replacing the  $B_p$ -seminorm by the  $A_p^1(\alpha)$ -one.

In both of these problems, we make use of a method based on two main ingredients:

- integral representation for the derivative of functions in  $K_\Theta$  or in  $\mathcal{R}_n^+$ , and
- a generalization of a theorem by E.M. Dyn'kin, see Subsection 2.2.3.

One more tool (that we will need in problem (2)) is the estimate of  $B_p$ -seminorms of finite Blaschke products by Arazy, Fischer and Peetre [AFP].

**2.2. Main results.** Let us consider the differentiation operator  $Df = f'$  and the shift and the backward shift operators defined respectively by

$$(2.2) \quad Sf = zf, \quad S^*f = \frac{f - f(0)}{z},$$

for any  $f \in \mathcal{H}ol(\mathbb{D})$ . From now on, for any inner function  $\Theta$ , we put

$$\tilde{\Theta} = z\Theta = S\Theta.$$

**2.2.1. Boundedness of the differentiation operator from  $(K_\Theta, \|\cdot\|_{BMOA})$  to  $A_p(\alpha)$ .** Let us first discuss the boundedness of the operator  $D$  from  $BMOA$  to  $A_p(\alpha)$ . The following (essentially well-known) proposition gives necessary and sufficient conditions on  $p$  and  $\alpha$  so that a continuous embedding  $BMOA \subset A_p^1(\alpha)$  hold.

**Proposition 2.1.** *Let  $\alpha > -1$  and  $1 \leq p < \infty$ . Then  $BMOA \subset A_p^1(\alpha)$  if and only if either  $\alpha > p - 1$  or  $\alpha = p - 1$  and  $p \geq 2$ .*

Now, we consider an arbitrary *inner function*  $\Theta$ . Our first main result gives necessary and sufficient conditions under which the differentiation operator

$$D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$$

is bounded. When this is the case, we estimate its norm in terms of  $\|\Theta'\|_{A_p(\alpha)}$ .

**Theorem 2.2.** *Let  $1 < p < \infty$  and  $\alpha > -1$ . Then the operator  $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$  is bounded if and only if  $\Theta' \in A_p(\alpha)$ .*

*Moreover, one can distinguish three cases:*

(a) *If  $\alpha > p-1$  or  $\alpha = p-1$ , and  $p \geq 2$  then the operator  $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$  is bounded.*

(b) *If  $p-2 < \alpha < p-1$ , then the operator  $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$  is bounded if and only if  $\Theta' \in A_1(\alpha - p + 1)$ .*

(c) *If  $\alpha \leq p-2$ , then the operator  $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$  is bounded if and only if  $\Theta$  is a finite Blaschke product.*

*In cases (b) and (c), we have*

$$(2.3) \quad \|D\| \lesssim \|\Theta'\|_{A_p(\alpha)} \lesssim \|D\| + \text{const},$$

*with constants depending on  $p$  and  $\alpha$  only.*

*Remark.* 1. In the cases (b) and (c), to show that their conditions are equivalent to the inclusion  $\Theta' \in A_p(\alpha)$  we use a theorem by P.R. Ahern [Ahe1] and its generalizations by I.E. Verbitsky [Ver] and A. Gluchoff [Glu]. We do not know whether the inclusions  $\Theta' \in A_p(\alpha)$  and  $\Theta' \in A_1(\alpha - p + 1)$  are equivalent for  $\alpha = p-1$  and  $1 < p < 2$ .

2. The membership of Blaschke products in various function spaces is a well-studied topic. Besides the above-cited papers by Ahern, Gluchoff and Verbitsky, let us mention the papers by Ahern and D.N. Clark [AC1, AC2] and recent works by D. Girela, J. Peláez, D. Vukotić, and A. Aleman [GPV, AV].

2.2.2. *Generalization of Peller's inequalities.* In the following theorem, we give a generalization of Peller's inequality (2.1).

**Theorem 2.3.** *Let  $f \in \mathcal{R}_n^+$ ,  $\deg f = n$  and  $\sigma \in \mathbb{D}^n$  be the set of its poles counting multiplicities (including poles at  $\infty$ ). For any  $\alpha > -1$ ,  $1 < p < \infty$ , and  $p > 1 + \alpha$ , we have*

$$(2.4) \quad \|f\|_{A_p^1(\alpha)}^* \leq K_{p,\alpha} \|f\|_{BMOA} \|B'_\sigma\|_{A_p(\alpha)},$$

where  $K_{p,\alpha}^p = \frac{2^{\frac{1}{\alpha+1}}}{2^{\frac{1}{\alpha+1}} - 1} \left( \frac{p}{p-1-\alpha} \right)^p 2^{p+1}$ .

*Remark.* The inequality (2.4) is sharp up to a constant in the following sense: for  $f = B_\sigma$  we, obviously, have  $\|f\|_{A_p^1(\alpha)}^* = \|f\|_{BMOA} \|B'_\sigma\|_{A_p(\alpha)}$  (note that  $\|B_\sigma\|_{BMOA} = 1$ ).

Let us show how Peller's inequality (2.1) for  $1 < p < \infty$  follows from Theorem 2.3. For  $\alpha = p-2$ , we have

$$\|f'\|_{A_p(\alpha)} = \|f\|_{B_p}^*, \quad \|B'_\sigma\|_{A_p(\alpha)} = \|B_\sigma\|_{B_p}^*.$$

To deduce Peller's inequalities it remains to apply the following theorem by Arazy, Fischer and Peetre [AFP]: if  $1 \leq p \leq \infty$ , then there exist absolute positive constants  $m_p$  and  $M_p$  such that

$$(2.5) \quad m_p n^{\frac{1}{p}} \leq \|B\|_{B_p}^* \leq M_p n^{\frac{1}{p}}.$$

for any Blaschke product of degree  $n$ . Then we obtain for  $1 < p < \infty$ ,

$$\|f\|_{B_p}^* \leq K_{p,p-2}^{\frac{1}{p}} M_p \|f\|_{BMOA} (n+1)^{\frac{1}{p}} \lesssim n^{\frac{1}{p}} \|f\|_{BMOA}.$$

To make the expositions self-contained, we give in Section 5 a very simple proof of the upper estimate in (2.5) (which is slightly different from the proof by D. Marshall presented in [AFP]).

The method of integral representations for higher order derivatives in model spaces allows to prove Peller's inequalities also for  $0 < p \leq 1$ . In Section 6 we present the proof for the case  $p > \frac{1}{2}$ .

**2.2.3. Generalization of a theorem by Dyn'kin.** E.M. Dyn'kin proved in [Dyn, Theorem 3.2] that

$$(2.6) \quad \int_{\mathbb{D}} \left( \frac{1 - |B(u)|^2}{1 - |u|^2} \right)^2 dA(u) \leq 8(n+1),$$

for any finite Blaschke product  $B$  of degree  $n$ .

From now on, for any inner function  $\Theta$  and for any  $\alpha > -1$ ,  $p > 1$ , we put

$$(2.7) \quad I_{p,\alpha}(\Theta) = \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left( \frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u).$$

Dyn'kin's Theorem can be stated as follows: *for any finite Blaschke product  $B$  of degree  $n$ , we have*

$$I_{2,0}(B) \leq 8(n+1).$$

Here, we generalize this result to the case  $\alpha > -1$ ,  $p > 1$  and  $p > 1 + \alpha$ . This generalization is the key step of the proof of Theorem 2.3.

**Theorem 2.4.** *Let  $1 < p < \infty$ ,  $\alpha > -1$  and  $p > 1 + \alpha$ . Then,*

$$\|\Theta'\|_{A_p(\alpha)}^p \leq I_{p,\alpha}(\Theta) \leq K_{p,\alpha} \|\Theta'\|_{A_p(\alpha)}^p,$$

where  $K_{p,\alpha}$  is the same constant as in Theorem 2.3.

The paper is organized as follows. We first focus in Section 3 on the generalization of Dyn'kin's result. In Section 4, Proposition 2.1 and Theorem 2.2 are proved, while Section 5 is devoted to the proof of Peller type inequalities (Theorem 2.3). The case  $\frac{1}{2} < p \leq 1$  in Peller's inequality is considered in Section 6. In Section 7, we discuss some estimates of radial-weighted Bergman norms of Blaschke products. Finally, in Section 8 we discuss some related inequalities by Dolzhenko for which we give a very simple proof for the case  $1 \leq p \leq 2$  based on Dyn'kin's estimate and suggest a way to extend these inequalities to the range  $p > 2$ .

### 3. GENERALIZATION OF DYN'KIN'S THEOREM

The aim of this Section is to prove Theorem 2.4. The lower bound follows trivially from the Schwarz–Pick inequality applied to  $\Theta$ . The main ideas for the proof of the upper bound come from [Dyn, Theorem 3.2]. In this Section,  $\Theta$  is an arbitrary inner function.

**Lemma 3.1.** *For  $p > 1$ ,  $\alpha > -1$  and  $p > 1 + \alpha$ , we have*

$$I_{p,\alpha}(\Theta) \leq 2^p \int_0^1 \int_0^{2\pi} (1-r)^\alpha \left( \frac{1}{1-r} \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \frac{d\theta}{\pi}.$$

*Proof.* Writing the integral  $I_{p,\alpha}(\Theta)$  in polar coordinates, and using the fact that

$$1 - |\Theta(u)|^2 \leq 2(1 - |\Theta(u)|),$$

we obtain

$$\begin{aligned} I_{p,\alpha}(\Theta) &\leq 2^p \int_0^1 r(1-r^2)^{\alpha-p} \left( \int_0^{2\pi} (1 - |\Theta(re^{i\theta})|)^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 r(1-r)^{\alpha-p} \left( \int_0^{2\pi} |\Theta(e^{i\theta}) - \Theta(re^{i\theta})|^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 r(1-r)^{\alpha-p} \left( \int_0^{2\pi} \left( \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p \frac{d\theta}{\pi} \right) dr \\ &\leq 2^p \int_0^1 \int_0^{2\pi} (1-r)^\alpha \frac{1}{(1-r)^p} \left( \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \frac{d\theta}{\pi}, \end{aligned}$$

which completes the proof of the lemma.  $\square$

We recall now a general version of the Hardy inequality, see [HLP, page 245], which after change of variables gives (as in [Ahe2, Lemma 7]):

**Lemma 3.2.** *If  $h : (0, 1) \rightarrow [0, +\infty)$ ,  $p > 1$ ,  $\alpha > -1$  and  $p > 1 + \alpha$ , then*

$$\int_0^1 (1-r)^\alpha \left( \frac{1}{1-r} \int_r^1 h(s) ds \right)^p dr \leq \left( \frac{p}{p-1-\alpha} \right)^p \int_0^1 (1-r)^\alpha h(r)^p dr.$$

**Corollary 3.3.** *Let  $p > 1$ ,  $\alpha > -1$  and  $p > 1 + \alpha$ . Then,*

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi},$$

where  $C_{p,\alpha} = \left( \frac{p}{p-1-\alpha} \right)^p 2^p$ .

*Proof.* Combining estimates in Lemma 3.1 and Lemma 3.2 (setting  $h(s) = h_\theta(s) = |\Theta'(se^{i\theta})|$ , for any fixed  $\theta \in [0, 2\pi)$ ), we obtain

$$\int_0^1 (1-r)^\alpha \left( \frac{1}{1-r} \int_r^1 |\Theta'(se^{i\theta})| ds \right)^p dr \leq \left( \frac{p}{p-1-\alpha} \right)^p \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr.$$

Thus,

$$I_{p,\alpha}(\Theta) \leq \left( \frac{p}{p-1-\alpha} \right)^p 2^p \int_0^{2\pi} \int_0^1 (1-r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi}$$

which completes the proof.  $\square$

**Lemma 3.4.** *Let any nonzero weight  $w$  satisfying  $w \geq 0$  and  $\int_0^1 w(r) dr < \infty$ . Let  $\beta = \beta_w \in (0, 1)$  such that  $\int_0^1 w(r) dr = 2 \int_0^\beta w(r) dr$ . Then, for  $f \in A_p(w)$ ,  $1 \leq p < \infty$ ,*

$$\begin{aligned} \|f\|_{A_p(w)}^p &\leq \int_0^{2\pi} w(r) \int_0^1 |f(re^{i\theta})|^p dr \frac{d\theta}{\pi} \\ &\leq \frac{2}{\beta} \int_\beta^1 rw(r) \left( \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi} \right) dr \leq \frac{2}{\beta} \|f\|_{A_p(w)}^p. \end{aligned}$$

*Proof.* The proof follows easily from the fact that for any  $f$  in  $\mathcal{H}ol(\mathbb{D})$ , the function

$$r \mapsto \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{\pi},$$

is nondecreasing on  $[0, 1]$ . □

We are now ready to prove Theorem 2.4.

*Proof.* We first prove (2.4). Applying Lemma 3.4 with  $f = \Theta'$  and  $w(r) = (1 - r^2)^\alpha$ ,  $\alpha > -1$ , and Corollary 3.3 we obtain that

$$I_{p,\alpha}(\Theta) \leq C_{p,\alpha} \int_0^{2\pi} \int_0^1 (1 - r)^\alpha |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} \leq \frac{2}{\beta} C_{p,\alpha} \|\Theta'\|_{A_p(\alpha)}^p,$$

where  $C_{p,\alpha} = \left(\frac{p}{p-1-\alpha}\right)^p 2^p$ , and  $\beta = \beta_\alpha$  satisfies the condition

$$\int_\beta^1 w(r) dr = \int_0^\beta w(r) dr.$$

By a direct computation, we see that  $\beta = \beta_\alpha$  is given by the equation  $\frac{1-(1-\beta)^{\alpha+1}}{1+\alpha} = \frac{(1-\beta)^{\alpha+1}}{1+\alpha}$ , which is equivalent to

$$(3.1) \quad \beta = \beta_\alpha = 1 - \frac{1}{2^{\frac{1}{\alpha+1}}}.$$

□

#### 4. PROOF OF PROPOSITION 2.1 AND THEOREM 2.2

**4.1. Proof of Proposition 2.1.** The statement for  $\alpha > p - 1$  is trivial. Indeed, by the standard Cauchy formula,

$$f'(u) = \left\langle f, \frac{z}{(1 - \bar{u}z)^2} \right\rangle, \quad u \in \mathbb{D},$$

and thus, bounding  $|f'(u)|$  from above by  $\|f\|_{BMOA} \left\| \frac{z}{(1 - \bar{u}z)^2} \right\|_{H^1} = (1 - |u|^2)^{-1} \|f\|_{BMOA}$ , we get

$$\|f'\|_{A_p(\alpha)}^p \lesssim \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|)^{\alpha-p} dA(u) \lesssim \|f\|_{BMOA}^p.$$



For  $p \geq 2$  and  $\alpha = p - 1$  we have

$$\begin{aligned} \|f'\|_{A_p(\alpha)}^p &= \int_{\mathbb{D}} (1 - |u|^2) |f'(u)|^p dA(u) \\ &\leq \|f\|_{B_\infty}^{p-2} \int_{\mathbb{D}} (1 - |u|^2) |f'(u)|^2 dA(u), \end{aligned}$$

where  $\|f\|_{B_\infty}$  is the norm of  $f$  in the Bloch space. Since  $\int_{\mathbb{D}} (1 - |u|^2) |f'(u)|^2 dA(u) \leq \|f\|_{H^2}^2$ ,  $\|f\|_{H^2} \lesssim \|f\|_{BMOA}$  and  $\|f\|_{B_\infty} \lesssim \|f\|_{BMOA}$ , we conclude that

$$(4.1) \quad \|f'\|_{A_p(\alpha)} \lesssim \|f\|_{BMOA}.$$

Now we turn to the necessity of the restrictions on  $p$  and  $\alpha$  for the estimate (4.1). If  $\alpha < p - 1$  then it is well known that there exist interpolating Blaschke products  $B$  such that  $B' \notin A_p(\alpha)$  (see, e.g., [Glu, Theorem 6], where an explicit criterion for the inclusion is given in terms of the zeros of  $B$ ). Finally, by a result of S.A. Vinogradov [Vin, Lemma 1.6], if  $f \in A_p^1(p - 1)$ ,  $1 \leq p < 2$ , then,  $\sum_{n=0}^{\infty} |\hat{f}(2^n)|^p < \infty$  (where  $\hat{f}(n)$  stands for the  $n^{th}$  Taylor coefficient of  $f$ ). Hence,  $A_p^1(p - 1)$  does not contain even some functions from the disc algebra, and so  $BMOA \not\subseteq A_p^1(p - 1)$  when  $1 \leq p < 2$ .  $\square$

**4.2. Integral representation for the derivative of functions in  $K_\Theta$ .** An important ingredient of our proof is the following simple and well-known integral representation for the derivative of a function from a model space.

**Lemma 4.1.** *Let  $\Theta$  be an inner function,  $f \in K_\Theta$ ,  $n \in \mathbb{N}$ . We have*

$$f^{(n)}(u) = \left\langle f, z^n (k_u^\Theta)^{n+1} \right\rangle,$$

for any  $u \in \mathbb{D}$ .

*Proof.* For a fixed  $u \in \mathbb{D}$ , we have

$$f^{(n)}(u) = \left\langle f, \frac{z^n}{(1 - \bar{u}z)^{n+1}} \right\rangle = \left\langle f, z^n (k_u^\Theta)^{n+1} \right\rangle.$$

Here the first equality is the standard Cauchy formula, while the second follows from the fact that  $z^n(1 - \bar{u}z)^{-n-1} - z^n(k_u^\Theta(z))^{n+1} \in \Theta H^2$  and  $f \perp \Theta H^2$ .  $\square$

**4.3. Proof of the left-hand side inequality in (2.3).** Sufficiency of the condition  $\Theta' \in A_p(\alpha)$  in Theorem 2.2 and the left-hand side inequality in (2.3) follow immediately from Theorem 2.4 and the following proposition.

**Proposition 4.2.** *Let  $\alpha > -1$  and  $1 < p < \infty$ , let  $\Theta$  be an inner function and  $f \in K_\Theta$ . Then we have*

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} (I_{p,\alpha}(\Theta))^{\frac{1}{p}}.$$

*Proof.* We use the integral representation of  $f'$  from Lemma 4.1:

$$f'(u) = \left\langle f, z (k_u^\Theta)^2 \right\rangle = \int_{\mathbb{T}} f(\tau) \overline{\tau (k_u^\Theta(\tau))^2} dm(\tau),$$

for any  $u \in \mathbb{D}$ , and thus

$$\begin{aligned} \|f'\|_{A_p(\alpha)}^p &= \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left| \int_{\mathbb{T}} f(\tau) \overline{\tau (k_u^\Theta(\tau))^2} dm(\tau) \right|^p dA(u) \\ &\leq \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left( \int_{\mathbb{T}} |k_u^\Theta(\tau)|^2 dm(\tau) \right)^p dA(u) \\ &= \|f\|_{BMOA}^p \int_{\mathbb{D}} (1 - |u|^2)^\alpha \left( \frac{1 - |\Theta(u)|^2}{1 - |u|^2} \right)^p dA(u), \end{aligned}$$

which completes the proof.  $\square$

It remains to combine Proposition 4.2 with Theorem 2.4 to complete the proof of the left-hand side inequality in (2.3).

**4.4. Proof of the right-hand side inequality in (2.3).** To prove the necessity of the inclusion  $\Theta' \in A_p(\alpha)$  and the left-hand side inequality in (2.3), consider the test function

$$f = S^* \Theta = \frac{\Theta - \Theta(0)}{z},$$

where  $S^*$  is the backward shift operator (2.2). It is well-known that  $f$  belongs to  $K_\Theta$  and easy to check that  $\|f\|_{BMOA} \leq 2$ , whence

$$(4.2) \quad \|D\|_{(K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)} \geq \frac{\|f'\|_{A_p(\alpha)}}{2}.$$

Now,

$$\|f'\|_{A_p(\alpha)}^p \geq \int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} |f'(r\xi)|^p dm(\xi) dr,$$

where  $\beta_\alpha$  is given by (3.1) and thus,

$$\begin{aligned} \|f'\|_{A_p(\alpha)} &\geq \left( \int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr \right)^{\frac{1}{p}} \\ &\quad - \left( \int_{\beta_\alpha}^1 r w(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi) - \Theta(0)}{r^2 \xi^2} \right|^p dm(\xi) dr \right)^{\frac{1}{p}}. \end{aligned}$$

On one hand, applying Lemma 3.4 with  $w = w_\alpha$  and  $\beta = \beta_\alpha$  we obtain

$$\begin{aligned} \int_{\beta_\alpha}^1 r w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta'(r\xi)}{r\xi} \right|^p dm(\xi) dr &\geq \int_0^{2\pi} \int_{\beta_\alpha}^1 r w_\alpha(r) |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} \\ &\geq \frac{\beta_\alpha}{2} \int_0^{2\pi} \int_0^1 r w_\alpha(r) |\Theta'(re^{i\theta})|^p dr \frac{d\theta}{\pi} = \frac{\beta_\alpha}{2} \|\Theta'\|_{A_p(\alpha)}^p. \end{aligned}$$

On the other hand, since  $\|f\|_{H^\infty} \leq 2$ , we have

$$\int_{\beta_\alpha}^1 r w_\alpha(r) \int_{\mathbb{T}} \left| \frac{\Theta(r\xi) - \Theta(0)}{r^2 \xi^2} \right|^p dm(\xi) dr \leq 2^p \int_{\beta_\alpha}^1 \frac{w_\alpha(r)}{r^{p-1}} dr \leq \frac{2^p}{\beta_\alpha^{p-1}} \int_{\beta_\alpha}^1 w_\alpha(r) dr.$$

Finally, we conclude that

$$\|f'\|_{A_p(\alpha)} \geq \left(\frac{\beta_\alpha}{2}\right)^{\frac{1}{p}} \|\Theta'\|_{A_p(\alpha)} - 2\beta_\alpha^{\frac{1}{p}-1} \left(\int_{\beta_\alpha}^1 w_\alpha(r) dr\right)^{\frac{1}{p}},$$

which, combined with (4.2), gives us the right-hand side inequality in (2.3).  $\square$

**4.5. Proof of Theorem 2.2.** To complete the proof of Theorem 2.2, we need to recall the following theorem proved by Ahern [Ahe1] for the case  $1 \leq p \leq 2$  and generalized by Verbitsky [Ver] and Gluchoff [Glu] to the range  $1 \leq p < \infty$ . This theorem characterizes inner functions  $\Theta$  whose derivative belong to  $A_p(\alpha)$ .

**Theorem.** ([Glu]) *Let  $\Theta$  be an inner function,  $1 \leq p < \infty$ , and  $\alpha > -1$ .*

- (i) *If  $\alpha > p - 1$ , then  $\Theta' \in A_p(\alpha)$ .*
- (ii) *If  $p - 2 < \alpha < p - 1$ , then  $\Theta' \in A_p(\alpha)$  if and only if  $\Theta' \in A_1(\alpha - p + 1)$ .*
- (iii) *If  $\alpha < p - 2$  and  $p > 1$ , then  $\Theta' \in A_p(\alpha)$  if and only if  $\Theta$  is a finite Blaschke product.*

*Proof of Theorem 2.2.* Statement (a) is contained in Proposition 2.1. In order to prove (b) and (c) of Theorem 2.2, we first remark that for  $\alpha < p - 1$ , it follows from (2.3) that  $D : (K_\Theta, \|\cdot\|_{BMOA}) \rightarrow A_p(\alpha)$  is bounded if and only if  $\Theta' \in A_p(\alpha)$ . A direct application of the above Ahern–Verbitsky–Gluchoff theorem completes the proof for  $\alpha > p - 2$ . The case  $\alpha = p - 2$  follows from the Arazy–Fisher–Peetre inequality (2.5).  $\square$

## 5. PROOF OF PELLER TYPE INEQUALITIES

In this section we prove Theorem 2.3. From now on the inner function  $\Theta$  is a finite Blaschke product. Recall that if  $f \in \mathcal{R}_n^+$  and  $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$  are the poles of  $f$  (repeated according to multiplicities), then  $f \in K_{zB_\sigma}$  with  $\sigma = (\lambda_1, \dots, \lambda_n)$ .

We start with the proof of the upper bound in the Arazy–Fisher–Peetre inequality (2.5).

**Lemma 5.1.** *Let  $B$  be a finite Blaschke product with the zeros  $\{z_j\}_{j=1}^n$ . Then*

$$|B''(u)| \leq \sum_{j=1}^n \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^3} + \left( \frac{1 - |B(u)|}{1 - |u|} \right)^2, \quad u \in \mathbb{D}.$$

*Proof.* Let  $B = \prod_{j=1}^n b_{z_j}$ , where  $b_\lambda = \frac{|\lambda|}{\lambda} \cdot \frac{\lambda - z}{1 - \bar{\lambda}z}$ . Then it is easy to see that

$$(5.1) \quad |B''(u)| \leq \sum_{j=1}^n \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^3} \left| \frac{B(u)}{b_{z_j}(u)} \right| + 2 \sum_{1 \leq j < k \leq n} \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^2} \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \left| \frac{B(u)}{b_{z_j}(u)b_{z_k}(u)} \right|,$$

To estimate the second sum in (5.1), first we note that  $\frac{1 - |\lambda|^2}{2|1 - \bar{\lambda}u|^2} \leq \frac{1 - |b_\lambda(u)|}{1 - |u|} \leq \frac{2(1 - |\lambda|^2)}{|1 - \bar{\lambda}u|^2}$ . Let us introduce the notations  $B_j = \prod_{l=1}^{j-1} b_{z_l}$  (assuming  $B_1 \equiv 1$ ) and  $\widehat{B}_k = \prod_{l=k}^n b_{z_l}$ . Then

$$(5.2) \quad \frac{1 - |B(u)|}{1 - |u|} \asymp \sum_{j=1}^n |B_j(u)| \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^2} \asymp \sum_{k=1}^n |\widehat{B}_{k+1}(u)| \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

It follows from the estimate  $|B/(b_{z_j}b_{z_k})| \leq |B_j\widehat{B}_{k+1}|$  and (5.2) that

$$\begin{aligned} \sum_{1 \leq j < k \leq n} \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^2} \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \left| \frac{B(u)}{b_{z_j}(u)b_{z_k}(u)} \right| &\leq \left( \sum_{j=1}^n |B_j(u)| \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^2} \right) \times \\ &\times \left( \sum_{k=1}^n |\widehat{B}_{k+1}(u)| \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \right) \lesssim \left( \frac{1 - |B(u)|}{1 - |u|} \right)^2. \end{aligned}$$

□

Using Lemma 5.1, we first obtain the Arazy–Fisher–Peetre inequality for  $p = 1$ :

$$\|B\|_{B_1} \lesssim \int_{\mathbb{D}} |B''(u)| dA(u) \lesssim \sum_{j=1}^n \int_{\mathbb{D}} \frac{1 - |z_j|^2}{|1 - \bar{z}_j u|^3} dA(u) + I_{2,0}(B) \lesssim n.$$

We used Dyn’kin’s inequality (2.6) and the fact that, by the well-known estimates (see [HKZ, Theorem 1.7]), each integral in the above sum does not exceed some absolute constant, which does not depend on  $z_j$ . Finally, for  $1 < p < \infty$ , we have

$$\begin{aligned} \|B\|_{B_p}^{*p} &\asymp \int_{\mathbb{D}} |B''(u)|^p (1 - |u|^2)^{2p-2} dA(u) \\ &\leq \left( \sup_{u \in \mathbb{D}} |B''(u)| (1 - |u|^2)^2 \right)^{p-1} \int_{\mathbb{D}} |B''(u)| dA(u) \lesssim n, \end{aligned}$$

since  $\sup_{u \in \mathbb{D}} |f''(u)| (1 - |u|)^2 \leq 2\|f\|_{H^\infty}$ .

*Proof of Theorem 2.3.* Let  $f \in \mathcal{R}_n^+$ ; there exists  $\sigma \in \mathbb{D}^n$  such that  $f \in K_{\widetilde{B}_\sigma}$ ,  $\widetilde{B}_\sigma = zB_\sigma$ . Then, by Proposition 4.2 we have

$$\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA} \left( I_{p,\alpha}(\widetilde{B}_\sigma) \right)^{\frac{1}{p}}$$

for any  $\alpha > -1$  and  $1 < p < \infty$ . Now applying Theorem 2.4, we obtain  $\|f\|_{A_p^1(\alpha)}^* \leq K_{p,\alpha} \|f\|_{BMOA} \|B'_\sigma\|_{A_p(\alpha)}$ . Finally, note that

$$\|\widetilde{B}'_\sigma\|_{A_p(\alpha)} \leq \|zB'_\sigma\|_{A_p(\alpha)} + \|B_\sigma\|_{A_p(\alpha)} \lesssim \|B'_\sigma\|_{A_p(\alpha)}.$$

□

*Remark.* In Subsection 2.2.2, we have shown how to deduce Peller’s inequality (2.1) from Theorem 2.3 and the result of Arazy–Fischer–Peetre (2.5). Let us show that for  $p \geq 2$  one can give a very simple proof which uses only Proposition 4.2 and Dyn’kin’s estimate  $I_{2,0}(\widetilde{B}_\sigma) \leq 8(n+2)$ , where  $n = \deg B_\sigma$ . Indeed, in this case, we have

$$\begin{aligned} I_{p,p-2}(\widetilde{B}_\sigma) &= \int_{\mathbb{D}} (1 - |u|^2)^{p-2} \left( \frac{1 - |\widetilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^{p-2+2} dA(u) \\ &= \int_{\mathbb{D}} \left( 1 - |\widetilde{B}_\sigma(u)|^2 \right)^{p-2} \left( \frac{1 - |\widetilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^2 dA(u) \leq I_{2,0}(\widetilde{B}_\sigma). \end{aligned}$$

It remains to apply Proposition 4.2 with  $\alpha = p - 2$ .

6. AN ELEMENTARY PROOF OF PELLER'S INEQUALITY FOR  $p > \frac{1}{2}$ 

In this section we prove the inequality

$$(6.1) \quad \|f\|_{B_p} \leq cn^{\frac{1}{p}} \|f\|_{BMOA}$$

for  $1 \geq p > \frac{1}{2}$  using the integral representations of the derivatives in model spaces. It is well known and easy to see that, for  $p > \frac{1}{2}$ ,

$$\|f\|_{B_p}^p \asymp |f(0)|^p + |f'(0)|^p + |f''(0)|^p + \int_{\mathbb{D}} |f'''(u)|^p (1 - |u|^2)^{3p-2} dA(u).$$

Thus in what follows it is the last integral (which we denote  $\|f\|_{B_p}^{\star\star}$ ) that we will estimate.

Let  $\Theta$  be an inner function and let  $f \in K_{\Theta}$ . Then, by Lemma 4.1,  $|f'''(u)| = |\langle f, z^3(k_u^{\Theta})^4 \rangle| \leq \|k_u^{\Theta}\|_4^4 \|f\|_{BMOA}$ , and so

$$\|f\|_{B_p}^{\star\star} \leq \|f\|_{BMOA}^p \int_{\mathbb{D}} \|k_u^{\Theta}\|_4^{4p} (1 - |u|^2)^{3p-2} dA(u).$$

**Lemma 6.1.** *For any  $u \in \mathbb{D}$ ,*

$$\|k_u^{\Theta}\|_4^4 = \frac{(1 + |u|^2)(1 - |\Theta(u)|^4)}{(1 - |u|^2)^3} - \frac{4\operatorname{Re}(u\Theta'(u)\overline{\Theta(u)})}{(1 - |u|^2)^2}.$$

*Proof.* The lemma follows from straightforward computations based on the formula  $f'(u) = \langle f, z(1 - \bar{u}z)^{-2} \rangle$ . We omit the details.  $\square$

We continue to estimate  $\|f\|_{B_p}^{\star\star}$ . Since  $1 + |u|^2 \leq 2$  and  $1 - |\Theta(u)|^4 \leq 2(1 - |\Theta(u)|^2)$ , we have

$$\|k_u^{\Theta}\|_4^4 \leq \frac{4(1 - |\Theta(u)|^2)}{(1 - |u|^2)^3} - \frac{4\operatorname{Re}(u\Theta'(u)\overline{\Theta(u)})}{(1 - |u|^2)^2}.$$

From now on assume that  $\Theta$  is a finite Blaschke product  $B = \prod_{k=1}^n b_{z_k}$ . Then

$$\begin{aligned} uB'(u)\overline{B(u)} &= u|B(u)|^2 \frac{B'(u)}{B(u)} = u|B(u)|^2 \sum_{k=1}^n \left( \frac{1}{u - z_k} + \frac{\bar{z}_k}{1 - \bar{z}_k u} \right) \\ &= |B(u)|^2 \sum_{k=1}^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} + |B(u)|^2 \sum_{k=1}^n \frac{z_k(1 - |z_k|^2)(1 - |u|^2)}{|1 - \bar{z}_k u|^2(u - z_k)}. \end{aligned}$$

Denote the last term by  $S_1(u)$ . Since  $|B(u)| \leq |b_{z_k}(u)|$ , we have

$$|S_1(u)| \leq \sum_{k=1}^n \frac{(1 - |u|^2)(1 - |z_k|^2)}{|1 - \bar{z}_k u|^3},$$

whence (recall that  $p \leq 1$ )

$$\int_{\mathbb{D}} \frac{|S_1(u)|^p}{(1 - |u|^2)^p} (1 - |u|^2)^{3p-2} dA(u) \leq \sum_{k=1}^n \int_{\mathbb{D}} \frac{(1 - |z_k|^2)^p}{|1 - \bar{z}_k u|^{3p}} (1 - |u|^2)^{3p-2} dA(u) \lesssim n,$$

since, by [HKZ, Theorem 1.7], each integral in the above sum does not exceed some constant depending only on  $p$ , but not on  $z_k$ .

Thus, to prove (6.1), it remains to estimate the weighted area integral of the difference

$$S_2(u) = \frac{1 - |B(u)|^2}{(1 - |u|^2)^2} - \frac{|B(u)|^2}{1 - |u|^2} \sum_{k=1}^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

We use again the notations  $B_k = \prod_{l=1}^{k-1} b_{z_l}$  (assuming  $B_1 \equiv 1$ ) and  $\widehat{B}_k = \prod_{l=k}^n b_{z_l}$ . It is easy to see that

$$(6.2) \quad \frac{1 - |B(u)|^2}{1 - |u|^2} = \sum_{k=1}^n |B_k(u)|^2 \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

Hence,

$$\begin{aligned} S_2(u) &= \sum_{k=1}^n |B_k(u)|^2 \cdot \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \cdot \frac{1 - |\widehat{B}_k(u)|^2}{1 - |u|^2} \\ &= \sum_{k=1}^n \sum_{l=k}^n \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \cdot \frac{1 - |z_l|^2}{|1 - \bar{z}_l u|^2} \cdot |B_l(u)|^2. \end{aligned}$$

Note that, by a formula analogous to (6.2), but without squares,

$$\frac{1 - |B(u)|}{1 - |u|} = \sum_{k=1}^n |B_k(u)| \frac{1 - |b_{z_k}(u)|}{1 - |u|} \geq \frac{1}{2} \sum_{k=1}^n |B_k(u)| \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2}.$$

Hence,

$$4 \left( \frac{1 - |B(u)|}{1 - |u|} \right)^2 = \sum_{k=1}^n \sum_{l=1}^n |B_k(u)| \cdot |B_l(u)| \cdot \frac{1 - |z_k|^2}{|1 - \bar{z}_k u|^2} \cdot \frac{1 - |z_l|^2}{|1 - \bar{z}_l u|^2}.$$

Denote the last double sum by  $S_3(u)$ . Since  $|B_k B_l| \geq |B_l|^2$ ,  $l \geq k$ , we see that  $S_2(u) \leq S_3(u)$ . Now we have

$$\begin{aligned} \int_{\mathbb{D}} \frac{|S_2(u)|^p}{(1 - |u|^2)^p} (1 - |u|^2)^{3p-2} dA(u) &\leq 4^p \int_{\mathbb{D}} \left( \frac{1 - |B(u)|}{1 - |u|} \right)^{2p} (1 - |u|^2)^{2p-2} dA(u) \\ &\lesssim I_{2p, 2p-2}(B) \lesssim \|B'\|_{B_{2p}}^{2p} \lesssim n. \end{aligned}$$

Here we used Theorem 2.4 to estimate  $I_{2p, 2p-2}(B)$  (recall that  $2p > 1$ ) and the Arazy–Fisher–Peetre inequality (2.5).  $\square$

## 7. RADIAL-WEIGHTED BERGMAN NORMS OF THE DERIVATIVE OF FINITE BLASCHKE PRODUCTS

Again, let  $n \geq 1$ ,  $\sigma = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$  and let  $B_\sigma$  be the finite Blaschke product corresponding to  $\sigma$ . For any  $1 < p < \infty$  and  $\alpha > -1$ , we set

$$\varphi_n(p, \alpha) = \sup \left\{ \|B'_\sigma\|_{A_p(\alpha)} : \sigma \in \mathbb{D}^n \right\}.$$

Note that for any  $n \geq 1$ ,  $\varphi_n(p, \alpha) = \varphi_1(p, \alpha) = \infty$  if and only if  $\alpha < p - 2$ . Indeed, if  $\alpha \geq p - 2$ , then  $\varphi_n(p, \alpha) \leq \varphi_n(p, p - 2) \asymp n^{\frac{1}{p}}$  by the Arazy–Fisher–Peetre inequality (2.5). For  $\alpha < p - 2$ , consider the test function  $b_r$ ,  $r \in (0, 1)$ . It is easily seen (see, e.g., [HKZ, Theorem 1.7]) that  $\|b'_r\|_{A_p(\alpha)} \rightarrow \infty$  as  $r \rightarrow 1^-$ .

We have seen in Subsection 2.2.2 how the estimate  $\varphi_n(p, p-2) \asymp n^{\frac{1}{p}}$  implies Peller's inequality (2.1). It could be of interest to find a more general estimate (for other values of  $\alpha$  and  $p$ ) of  $\varphi_n(p, \alpha)$ . Notice that for each fixed  $p$ , the function  $\alpha \mapsto \varphi_n(p, \alpha)$  is decreasing and there exists the second critical value  $\alpha_p \geq -1$ ,

$$\alpha_p = \inf \left\{ \alpha > -1 : \sup_n \varphi_n(p, \alpha) < \infty \right\}.$$

The sequence  $\{\varphi_n(p, \alpha)\}_{n \geq 1}$  may be unbounded and, thus, have a nontrivial asymptotics if and only if  $p-2 \leq \alpha \leq \alpha_p$ . In this notation we can rewrite Theorem 2.3 as

$$\|f\|_{A_p^1(\alpha)}^* \lesssim \varphi_n(p, \alpha) \|f\|_{BMOA},$$

for any  $f \in \mathcal{R}_n^+$ ,  $1 < p < \infty$ ,  $p-2 \leq \alpha \leq \alpha_p$ .

We will show now that  $\alpha_p = p-1$ , and so  $p-1$  is the second critical value of  $\alpha$ , as is expected from Theorem 2.2.

**Proposition 7.1.** *For any  $p > 1$ ,  $\alpha_p = p-1$ .*

*Proof.* By the Schwarz–Pick lemma, we have that for any  $\sigma \in \mathbb{D}^n$ ,

$$\|B'_\sigma\|_{A_p(\alpha)}^p \leq I_{p,\alpha}(B_\sigma),$$

and for any  $\alpha > p-1$ ,

$$I_{p,\alpha}(B_\sigma) \leq \int_{\mathbb{D}} (1-|u|^2)^\alpha \left( \frac{1-|B_\sigma(u)|^2}{1-|u|^2} \right)^p dA(u) \lesssim \int_{\mathbb{D}} \frac{1}{(1-|u|)^{p-\alpha}} dA(u) < \infty,$$

and thus  $\alpha_p \leq p-1$  for each  $p$ .

Next we show that  $\alpha_p \geq p-1$ . Let us consider the set  $\sigma = (0, \dots, 0) \in \mathbb{D}^n$ , for which  $B_\sigma(z) = z^n$ . In this case, we have

$$\|B'_\sigma\|_{A_p(\alpha)}^p = \|nz^{n-1}\|_{A_p(\alpha)}^p = n^p \int_0^1 r(1-r^2)^\alpha \int_{\mathbb{T}} |r\xi|^{p(n-1)} dm(\xi) dr,$$

which gives

$$\|B'_\sigma\|_{A_p(\alpha)}^p = n^p \int_0^1 (1-r^2)^\alpha r^{p(n-1)+1} dr,$$

and

$$\beta(pn-p+2, \alpha+1) \leq \frac{\|B'_\sigma\|_{A_p(\alpha)}^p}{n^p} \leq 2^\alpha \beta(pn-p+2, \alpha+1),$$

where  $\beta$  stands for the *Beta function*  $\beta(x, y) = \int_0^1 r^{x-1}(1-r)^{y-1} dr$ . Let  $\alpha = p-1-\varepsilon$ ,  $\varepsilon > 0$ . Then by the standard  $\Gamma$ -function asymptotics, we obtain

$$\|B'_\sigma\|_{A_p(\alpha)}^p \geq \Gamma(\alpha+1)n^p \frac{\Gamma(pn-p+2)}{\Gamma(pn+\alpha-p+3)} \sim_{n \rightarrow \infty} \Gamma(\alpha+1)n^p(pn)^\varepsilon,$$

whence  $\sup_n \varphi_p(\alpha, n) = \infty$ . □

## 8. REMARKS ON DOLZHENKO'S INEQUALITIES

**8.1. Proof of Dolzhenko's inequalities for  $1 \leq p \leq 2$ .** In [Dol, Theorem 2.2] E.P. Dolzhenko proved that for any  $f \in \mathcal{R}_n^+$ ,

$$(8.1) \quad \|f'\|_{A_p} \lesssim \begin{cases} n^{1-\frac{1}{p}} \|f\|_{H^\infty}, & 1 < p \leq 2, \\ \log n \|f\|_{H^\infty}, & p = 1, \end{cases}$$

where the constants involved in  $\lesssim$  may depend on  $p$  only. Let us show that these inequalities (and even with  $BMOA$ -norm in place of  $H^\infty$ -norm) are direct corollaries of Proposition 4.2 and the following simple lemma.

**Lemma 8.1.** *For any Blaschke product  $B$  of degree  $n$  we have*

$$(8.2) \quad I_{p,0}(B) = \int_{\mathbb{D}} \left( \frac{1 - |B(u)|^2}{1 - |u|^2} \right)^p dA(u) \lesssim \begin{cases} n^{p-1}, & 1 < p \leq 2, \\ \log n, & p = 1. \end{cases}$$

*Proof.* Clearly, the integral over the disc  $\{|z| \leq 1 - \frac{1}{n}\}$  has the required estimate. The estimate over the annulus  $\{1 - \frac{1}{n} \leq |z| < 1\}$  follows from the result of Dyn'kin ( $I_{2,0}(B) \lesssim n$ ) and the Hölder inequality. Indeed, for  $1 \leq p < 2$ ,

$$\int_{\{1-\frac{1}{n} \leq |z| < 1\}} \left( \frac{1 - |B(u)|^2}{1 - |u|^2} \right)^p dA(u) \leq (I_{2,0}(B))^{\frac{p}{2}} \left( \pi \left( 1 - \left( 1 - \frac{1}{n} \right)^2 \right) \right)^{1-\frac{p}{2}} \lesssim n^{p-1}.$$

□

Now inequality (8.1) follows from (8.2) and from the inequality  $\|f'\|_{A_p(\alpha)} \leq \|f\|_{BMOA(I_{p,\alpha}(B_\sigma))}^{\frac{1}{p}}$  which holds for any function  $f \in K_{B_\sigma}$  (see Proposition 4.2). It should be mentioned, however, that Dolzhenko proves his inequalities for more general domains than the unit disc.

**8.2. An extension of Dolzhenko's inequalities to the range  $p > 2$ .** The case  $p > 2$  is also treated by Dolzhenko (see the last inequality in [Dol, Theorem 2.2]), but the corresponding analog is of somewhat different nature. As the example  $f(z) = (1 - rz)^{-1}$  with  $r \rightarrow 1-$  shows, there exist no estimate of  $\|f'\|_{A^p}$  in terms of  $\|f\|_{BMOA}$  and  $n = \deg f$ .

Here we obtain another extension of Dolzhenko's result for  $p > 2$ .

**Theorem 8.2.** *Let  $2 < p \leq \infty$ , let  $f \in \mathcal{R}_n^+$ ,  $n \geq 1$ , and let  $1/\bar{\lambda}_1, \dots, 1/\bar{\lambda}_n$  be its poles (repeated according to multiplicities). We have*

$$(8.3) \quad \|f'\|_{A_p} \lesssim n^{\frac{1}{p}} \left( \sum_{k=1}^n \frac{1 + |\lambda_k|}{1 - |\lambda_k|} \right)^{1-\frac{2}{p}} \|f\|_{BMOA}.$$

Moreover, the inequality (8.3) is asymptotically sharp in the following sense: for any  $r \in (0, 1)$  there exists  $g \in \mathcal{R}_n^+$  having  $\frac{1}{r}$  as a pole of multiplicity  $n$  such that

$$(8.4) \quad \|g'\|_{A_p} \gtrsim n^{1-\frac{1}{p}} \left( \frac{1}{1-r} \right)^{1-\frac{2}{p}} \|g\|_{BMOA}.$$



*Proof.* We first prove (8.3). Set  $\sigma = (\lambda_1, \dots, \lambda_n)$ , so that  $f \in K_{\tilde{B}_\sigma}$ . By Proposition 4.2 and Dyn'kin's inequality (2.6),

$$\|f'\|_{A_p}^p \leq \|f\|_{BMOA}^p \int_{\mathbb{D}} \left( \frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^p dA(u) \lesssim n \|f\|_{BMOA} \sup_{u \in \mathbb{D}} \left( \frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \right)^{p-2}.$$

By (6.2), we have

$$\frac{1 - |\tilde{B}_\sigma(u)|^2}{1 - |u|^2} \leq 1 + \sum_{k=1}^n \frac{1 - |\lambda_k|^2}{|1 - \overline{\lambda_k}u|^2} \lesssim \sum_{k=1}^n \frac{1 + |\lambda_k|}{1 - |\lambda_k|}.$$

This proves (8.3). Now we prove (8.4). Take  $g = b_{-r}^n(u)$ ,  $r \in (0, 1)$ , then

$$\|g'\|_{A_p}^p = n^p \int_{\mathbb{D}} |b'_{-r}(u)|^2 |b'_{-r}(u)|^{p-2} |b_{-r}(u)|^{p(n-1)} dA(u).$$

Taking  $v = b_{-r}(u)$  as the new variable and using the fact that  $u = b_{-r}(v)$ , we get

$$\|g'\|_{A_p}^p = n^p \int_{\mathbb{D}} |b'_{-r}(b_{-r}(v))|^{p-2} |v|^{p(n-1)} dA(v).$$

Since  $b'_{-r} \circ b_{-r}(v) = -\frac{(1+rv)^2}{1-r^2}$ , we obtain

$$\|g'\|_{A_p}^p = \frac{n^p}{(1-r^2)^{p-2}} \int_{\mathbb{D}} |1+rv|^{2(p-2)} |v|^{p(n-1)} dA(v).$$

Supposing that  $p \geq 2$ , we have

$$\int_{\mathbb{D}} |1+rv|^{2(p-2)} |v|^{p(n-1)} dA(v) \asymp \int_{\mathbb{D}} |v|^{p(n-1)} dA(v) = \frac{2}{pn - p + 2},$$

whence

$$\|g'\|_{A_p}^p \asymp \frac{n^{p-1}}{(1-r)^{p-2}}.$$

Since  $\|g\|_{BMOA} = 1$  this completes the proof (8.4).  $\square$

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